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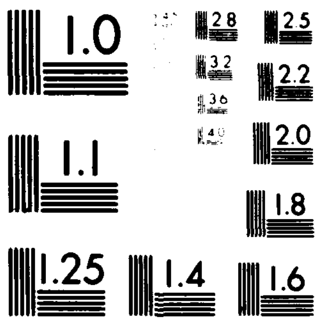

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SOME RESULTS ON THE OVERALL RELIABILITY

OF UNDIRECTED GRAPHS

by

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ABSTRACT

A probabilistic graph consists of vertices and links that fail with some known probabilities. For such a graph, *overall reliability* is the probability that there exists communication between all vertex-pairs. In this paper, some useful results are presented to simplify the overall reliability computation of an undirected graph when the failure events of the links are statistically independent.

Some Results on the Overall Reliability of Undirected Graphs

by

A. Satyanarayana, M. Chang and Z. Khalil

1. Introduction

A probabilistic graph consists of a set of vertices and links that fail with some known probability distribution. In reliability analysis of such a graph, one of the problems of considerable interest is the *overall reliability*. For a probabilistic graph, overall reliability is the probability that there exists communication between all vertex-pairs.

Wing and Demetriou [1] studied this problem by considering all possible vertex-pairs in the graph but the approach is not tractable for large graphs. Fu [2] provided approximate solutions using the well known techniques of electrical network analysis. Ball and Van Slyke [3] proposed a method of enumerating modified cutsets by backtracking. Satyanarayana [4], Satyanarayana and Hagstrom [5] made topological observations and proposed subgraph enumeration algorithms to compute the overall reliability. However all these methods have worst case computation time that grows exponentially in the size of the graph. Furthermore, this problem is believed to possess no algorithms with a polynomial time bound [6]. Therefore, it is essential to explore the use of graph partitioning techniques for efficient computation of the overall reliability. Graph partitioning may involve splitting the graph into reduced graphs, each analyzed separately and then the results combined. When the computational complexity grows more than linearly in the size of the graph, substantial savings may be achieved using reduced graphs instead of the given large graph.

In this paper, we present some results concerning the overall reliability of an undirected

graph in which the failure events of links are assumed to be statistically independent. Further ramifications of these results, the algorithmic implementation and computational complexity will be discussed elsewhere [7]. Suppose G is an undirected graph and m is a vertex of degree $k>1$ in G. In section 2, we present a scheme for decomposing G into $\binom{k}{2}+1$ reduced graphs, and express the reliability of G in terms of these reduced graphs. In section 3, we present two simple results which are useful in expressing the reliability of a biconnected graph in terms of the reliabilities of its subgraphs.

2. Overall Reliability of an Undirected Graph

One of the earliest decomposition techniques is that of *factoring* the graph with respect to a particular link [8,9]. For a given undirected graph G, two reduced graphs $G-e$ and G_e are obtained by deleting and contracting respectively a link e in G. Denote by $R(G)$ the overall reliability of G, and Pr{e} by p_e.

Property 1:

$$R(G) = p_e R(G_e) + (1-p_e) R(G-e).$$

Consider the example graph G of Fig.1. Using Property 1, the overall reliability

$$R(G) = p_3(p_1+p_2-p_1p_2)+(1-p_3)(p_1p_2).$$

For a complete solution of R(G) when G is larger, Property 1 may be repeatedly applied to the reduced graphs. Because of the binary structure of this procedure, the computation tree grows exponentially in the size of G. Simple techniques for reducing the size of the computation tree are therefore important. Besides the obvious savings obtained by not factoring on essential or irrelevant links, replacement of parallel links by a single link with the appropriate probability is another simple reduction [9]. Ball [10] and Johnson [11] independently have shown that the complexity of an algorithm based on repeated application of Property 1 and performing parallel reductions is of order (n-1)! for a complete graph on n vertices. This algorithm in [10,11] is perhaps the best known for computing overall reliability.

In what follows, we present a decomposition scheme in which the graph is factored with

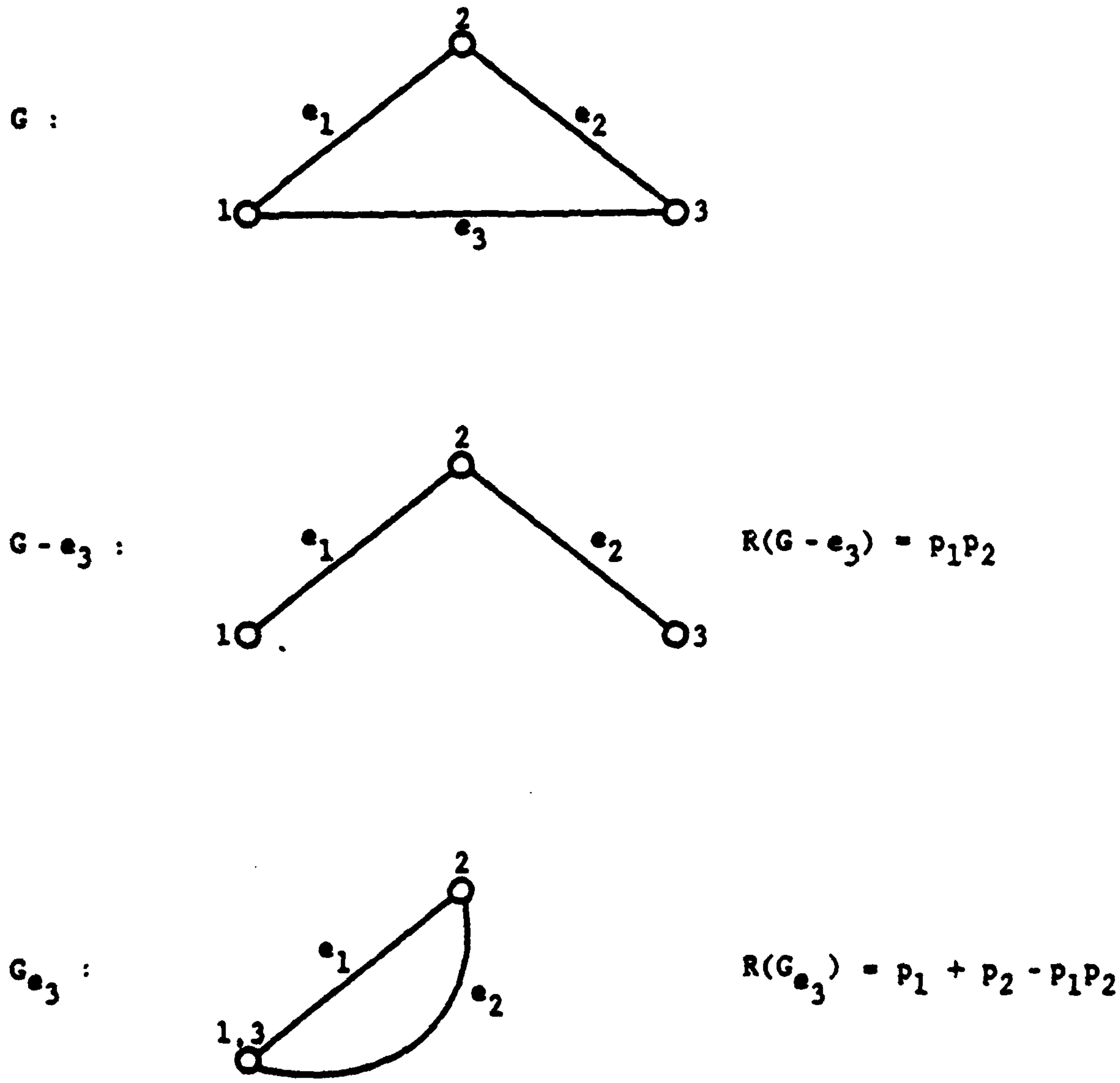


FIGURE 1

Example Graphs G, $G - e$ and G_e

respect to a particular vertex. This scheme results in a better computational bound than in [10,11]. In section 3 we introduce another decomposition technique to be applied on a biconnected graph.

Let m be a vertex of degree $k>1$ in G such that $e_1, e_2, ..., e_k$ are the links incident on m. Also, without loss of generality generality, let $K = \{1, 2, ..., k\}$ be the other end vertices of links $e_1, e_2, ..., e_k$. Denote by $G-m$ the subgraph obtained by deleting vertex m and its incident links from G. Let $X \subseteq K$ and G_X be the reduced graph obtained by coalescing vertices of X in $G-m$. We have the following lemma:

Lemma 1:

$$R(G) = \{\sum_{i=1}^{k} p_i \prod_{j \neq i} (1-p_j)\} R(G-m) + \sum_{\substack{X \subseteq K \\ |X|>1}} \prod_{i \in X} p_i \prod_{j \in K-X} (1-p_j)\, R(G_X),$$

where $p_j = Pr\{e_j\}$.

Proof: By deleting and contracting links $e_1, e_2, ..., e_k$ successively from vertex m in G, and applying Property 1 repeatedly on the resulting graphs, we can easily show that

$$R(G) = \sum_{\substack{X \subseteq K \\ X \neq \Phi}} \prod_{i \in X} p_i \prod_{j \in K-X} (1-p_j)\, R(G_X).$$

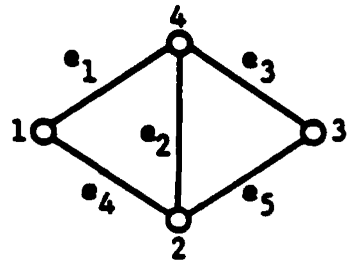
Since $G_X = G-m$ for $|X| = 1$, we have the lemma. QED.

To illustrate Lemma 1, consider the example G of Fig.2. We factor G on vertex 4. Vertices 1,2 and 3 are adjacent to 4. $G-m$ and all possible graphs obtained from $G-m$ by coalescing 1,2,3 in all possible combinations taking two vertices and three vertices at a time are shown in Fig.2. By Lemma 1 the overall reliability of G can be written as

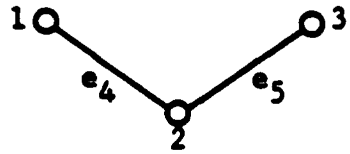
$$\begin{aligned} R(G) = &\{p_1(1-p_2)(1-p_3)+p_2(1-p_1)(1-p_3)+p_3(1-p_1)(1-p_2)\}R(G-m) \\ &+p_1p_2(1-p_3)R(G_{\{1,2\}})+p_1p_3(1-p_2)R(G_{\{1,3\}})+p_2p_3(1-p_1)R(G_{\{2,3\}}) \\ &+p_1p_2p_3R(G_{\{1,2,3\}}) \\ = &\, p_4p_5\{p_1(1-p_2)(1-p_3)+p_2(1-p_1)(1-p_3)+p_3(1-p_1)(1-p_2)\}+p_5\{p_1p_2(1-p_3)\} \\ &+(p_4+p_5-p_4p_5)\{p_1p_3(1-p_2)\}+p_4\{p_2p_3(1-p_1)\}+p_1p_2p_3. \end{aligned}$$

Clearly, for a given vertex m of degree k, the number of possible nonempty subsets X is 2^k-1. This is the number of reduced graphs obtained if G and the reduced graphs are

G :

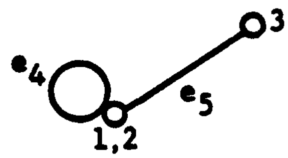


G - 4 :



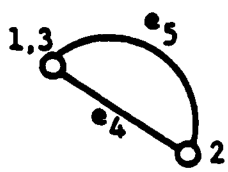
$R(G - 4) = p_4 p_5$

$G_{\{1,2\}}$



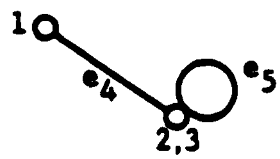
$R(G_{\{1,2\}}) = p_5$

$G_{\{1,3\}}$



$R(G_{\{1,3\}}) = p_4 + p_5 - p_4 p_5$

$G_{\{2,3\}}$



$R(G_{\{2,3\}}) = p_4$

$G_{\{1,2,3\}}$

$R(G_{\{1,2,3\}}) = 1$

FIGURE 2

Illustration for Lemma 1

successively factored w.r.t. links $e_1, e_2, \ldots, e_k$ using Property 1 until none of them contain these links. Among the 2^k-1 graphs, $k-1$ of them correspond to $G-m$. Lemma 1 recognizes this fact and the number of reduced graphs obtained by factoring G on m is 2^k-k. For computing the overall reliability of G, application of Lemma 1 thus provides a significant computational advantage over Property 1. In fact, we can do even better when we realize that the 2^k-k-1 terms in the second sum correspond to the partition of the event that at least two links out of $\{e_1, e_2, \ldots, e_k\}$ are functional. By the following theorem, we show that the number of reduced graphs need to be considered is $\binom{k}{2}+1$.

Theorem 1: The overall reliability of G,

$$R(G) = \left[\sum_{i=1}^{k} p_i \prod_{j \neq i} (1-p_j)\right] R(G-m) + \sum_{i<j} p_i p_j \prod_{i \neq q < j} (1-p_q)\, R(\hat{G}_{\{i,j\}}),$$

where $\hat{G}_{\{i,j\}}$ is the graph obtained from G by contracting links e_i and e_j (i<j) and deleting links e_q, ($i \neq q < j$).

Proof: Let ξ_1 and ξ_{2+} be the events that out of $\{e_1, e_2, \ldots, e_k\}$, exactly one link functions and at least two links function, respectively. Then,

$$R(G) = Pr\{\textit{all vertices in G are connected} \cap \xi_1\} + Pr\{\textit{all vertices in G are connected} \cap \xi_{2+}\}.$$

We know $Pr\{\xi_1\} = \sum_i p_i \prod_{j \neq i} (1-p_i)$. For ξ_{2+} to occur, one or more of the pairs $\{e_i, e_j\}$, $i<j$, must be functional. Applying the formula $Pr\{A \cup B\} = Pr\{A\} + Pr\{B \cap \bar{A}\}$ on the lexicographic ordering of these link pairs, we obtain a partition of ξ_{2+}:

$$Pr\{\xi_{2+}\} = \sum_{i<j} p_i p_j \prod_{i \neq q < j} (1-p_q).$$

Conditioning on this partition and that for ξ_1, we have

$$R(G) = \sum_i p_i \prod_{j \neq i} (1-p_j)\, Pr\{\textit{all vertices in G are connected} \mid e_i \textit{ functions but not } e_j,\ j \neq i\}$$

$$+ \sum_{i<j} p_i p_j \prod_{i \neq q < j} (1-p_q)\, Pr\{\textit{all vertices in G are connected} \mid e_i \text{ and } e_j \textit{ function}$$

$$\textit{but not } e_q,\ i \neq q < j\}$$

$$=\sum_i p_i \prod_{j\neq i}(1-p_j)\, R(G-m) + \sum_{i<j} p_i p_j \prod_{i\neq q<j} (1-p_q)\, R(\hat{G}_{\{i,j\}}).\ QED.$$

We now illustrate Theorem 1 using the graph of Fig.2. Reduced graphs $(G-m)$, $\hat{G}_{\{1,2\}}$, $\hat{G}_{\{1,3\}}$ and $\hat{G}_{\{2,3\}}$ obtained by factoring G on vertex 4 are shown in Fig.3. By Theorem 1 we have

$$\begin{aligned} R(G) = &\{p_1(1-p_2)(1-p_3) + p_2(1-p_1)(1-p_3) + p_3(1-p_1)(1-p_2)\}\, R(G-m) \\ &+ p_1p_2\, R(\hat{G}_{\{1,2\}}) + p_1p_3(1-p_2)\, R(\hat{G}_{\{1,3\}}) + p_2p_3(1-p_1)\, R(\hat{G}_{\{2,3\}}) \\ &= p_4p_5\, \{p_1(1-p_2)(1-p_3) + p_2(1-p_1)(1-p_3) + p_3(1-p_1)(1-p_2)\} \\ &+ (p_3 + p_5 - p_3p_5)p_1p_2 + (p_4 + p_5 - p_4p_5)p_1p_3(1-p_2) + p_4p_2p_3(1-p_1). \end{aligned}$$

3. Overall Reliability of a Biconnected Graph

For a connected graph G, a vertex a is a *cut-vertex* if removing a splits G into two or more parts. Suppose G_1, G_2, ..., G_k are the subgraphs of G which partition the links of G and furthermore, have a cut-vertex of G in common among them. The overall reliability of G can then be easily expressed as a product of the overall reliabilities of G_1, G_2, ..., G_k.

G is *biconnected* if it contains no cut-vertices, but there exists a pair of vertices whose deletion disconnects G. To emphasize the fact that deletion of a pair of vertices disconnects G, it is sometimes referred to as a *strictly biconnected* graph. Suppose G_1 and G_2 are the two subgraphs of G such that $G_1 \bigcup G_2 = G$ and $G_1 \bigcap G_2$ contains only two vertices i and j. Let $\hat{G}_1$ and $\hat{G}_2$ denote the graphs obtained by coalescing i and j in G_1 and G_2 respectively. We have the following theorem, which may be viewed as a generalization of Property 1.

Theorem 2: The overall reliability of G,

$$R(G) = R(G_1)R(\hat{G}_2) + \{R(\hat{G}_1) - R(G_1)\}R(G_2).$$

Proof: The proof is by induction on the number of links in G_2.

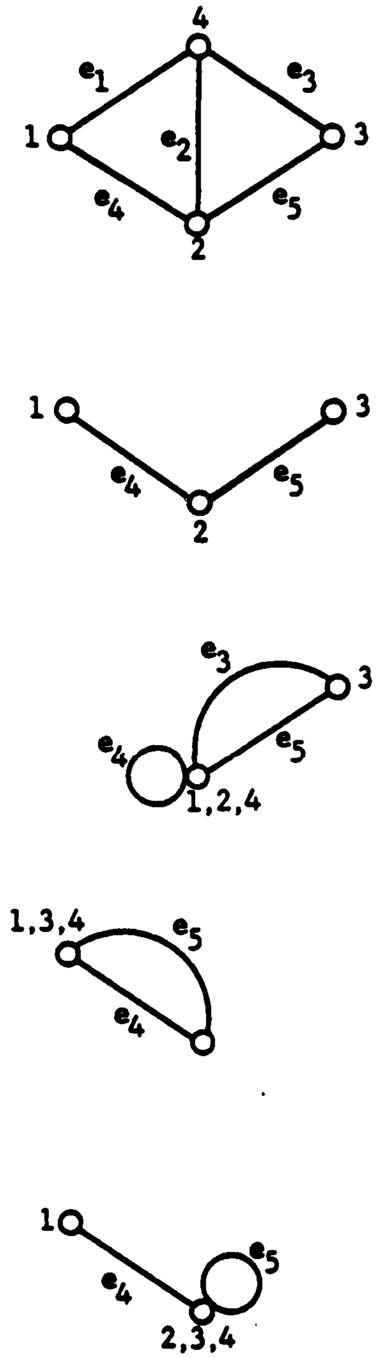
Suppose G_2 has exactly one link e. Using Property 1 on e we have

$$R(G) = p_e R(\hat{G}_1) + (1-p_e)R(G_1).$$

Since $R(G_2) = p_e$ and $R(\hat{G}_2) = 1$, we have

$$R(G) = R(G_1)R(\hat{G}_2) + R(G_2)R(\hat{G}_1) - R(G_1)R(G_2),$$

and the theorem is true.



G :

$G - 4$: $R(G - 4) = p_4p_5$

$\hat{G}_{\{1,2\}}$: $R(\hat{G}_{\{1,2\}}) = p_3 + p_5 - p_3p_5$

$\hat{G}_{\{1,3\}}$: $R(\hat{G}_{\{1,3\}}) = p_4 + p_5 - p_4p_5$

$\hat{G}_{\{2,3\}}$: $R(\hat{G}_{\{2,3\}}) = p_4$

FIGURE 3

Illustration for Theorem 1

Suppose the theorem is true for any G_2 with k links. Consider a graph G such that its G_2 has k+1 links. Let e be a link in G_2 and G_e be the graph obtained by contracting e in G. By Property 1,

$$R(G) = p_e R(G_e) + (1-p_e) R(G-e).$$

Note that deletion of i and j still disconnects G_e and of course $G-e$. Let G_e be partitioned as G_1 and G_{2_e} w.r.t. vertices i and j, where G_{2_e} is identical to the graph obtained from G_2 by contracting e. G_{2_e} has k links and hence the theorem is true for G_e and we have

$$R(G_e) = R(G_1)R(\hat{G}_2) + R(G_{2_e})R(\hat{G}_1) - R(G_1)R(G_{2_e}),$$

where $\hat{G}_2$ is the graph obtained from G_{2_e} by coalescing i and j. Similarly, $G-e$ can be partitioned w.r.t. i and j and let the subgraphs be G_1 and G_2-e. Here again G_2-e has k links and hence we have

$$R(G-e) = R(G_1)R(\bar{G}_2) + R(G_2-e)R(\hat{G}_1) - R(G_1)R(G_2-e),$$

where $\bar{G}_2$ is the graph obtained from G_2-e by coalescing i and j. Therefore,

$$\begin{aligned} R(G) &= p_e\{R(G_1)R(\hat{G}_2) + R(G_{2_e})R(\hat{G}_1) - R(G_1)R(G_{2_e})\} \\ &\quad + (1-p_e)\{R(G_1)R(\bar{G}_2) + R(G_2-e)R(\hat{G}_1) - R(G_1)R(G_2-e)\} \\ &= R(G_1)\{p_e R(\hat{G}_2) + (1-p_e)R(\bar{G}_2)\} + R(\hat{G}_1)\{p_e R(G_{2_e}) + (1-p_e)R(G_2-e)\} \\ &\quad - R(G_1)\{p_e R(G_{2_e}) + (1-p_e)R(G_2-e)\}. \end{aligned}$$

By Property 1, the above reduces to

$$R(G) = R(G_1)R(\hat{G}_2) + R(G_2)R(\hat{G}_1) - R(G_1)R(G_2).$$

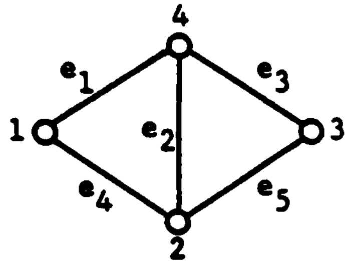
By induction on k, the theorem is true for all G. QED.

Theorem 2 is illustrated using the example graph of Fig.2. G_1, $\hat{G}_1$, G_2 and $\hat{G}_2$ are shown in Fig.4. By Theorem 2,

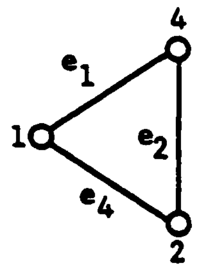
$$\begin{aligned} R(G) &= (p_1p_2 + p_1p_4 + p_2p_4 - 2p_1p_2p_4)(p_3 + p_5 - p_3p_5) + p_3p_5(p_1 + p_4 - p_1p_4) \\ &\quad - (p_1p_2 + p_1p_4 + p_2p_4 - 2p_1p_2p_4)p_3p_5. \end{aligned}$$

We have seen that by means of Theorem 2, the overall reliability of $G = G_1 \bigcup G_2$ can be expressed in terms of the overall reliability of its two subgraphs and their coalesced graphs ($\hat{G}_1$ and $\hat{G}_2$). In addition to the property that deletion of a pair of vertices i and j disconnects G,

G :

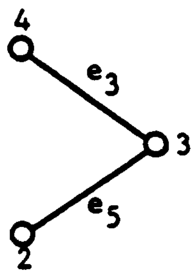


G_1 :



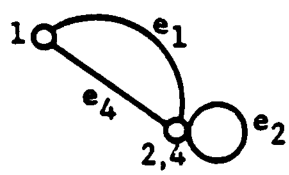
$R(G_1) = p_1p_2 + p_1p_4 + p_2p_4 - 2p_1p_2p_4$

G_2 :



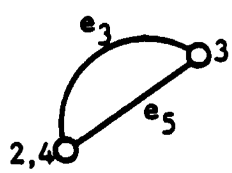
$R(G_2) = p_3p_5$

$\tilde{G}_1$:



$R(\tilde{G}_1) = p_1 + p_4 - p_1p_4$

$\tilde{G}_2$:



$R(\tilde{G}_2) = p_3 + p_5 - p_3p_5$

FIGURE 4

Illustration for Theorem 2

suppose $G_1 \bigcap G_2$ contains a link e between i and j. $R(G)$ can then be expressed in terms of G_1, G_2, G_1-e and G_2-e using the following theorem.

Theorem 3:

$$R(G) = \frac{1}{p_e}\{R(G_1)R(G_2) - (1-p_e)R(G_1-e)R(G_2-e)\}.$$

Proof: Let G_{1_e} and G_{2_e} be the graphs obtained by contracting e in G_1 and G_2 respectively. Since $G_1 \bigcup (G_2-e) = G$ and $G_1 \bigcap (G_2-e) = \{i, j\}$, the conditions of Theorem 2 are satisfied and we have

$$R(G) = R(G_1)R(G_{2_e}) + R(G_2-e)\{R(G_{1_e}) - R(G_1)\}.$$

By Property 1, $R(G_1) = p_e R(G_{1_e}) + (1-p_e)R(G_1-e)$, which implies

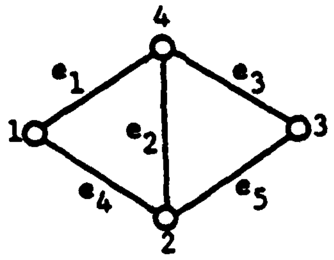
$$R(G_{1_e}) = \frac{1}{p_e}\{R(G_1) - (1-p_e)R(G_1-e)\},\ \ p_e \neq 0.$$

Similarly for $R(G_{2_e})$. Substituting the above for $R(G_{1_e})$ and $R(G_{2_e})$ in the expression for $R(G)$ yields the theorem. QED.

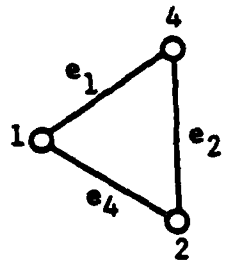
Theorem 3, which expresses $R(G)$ in terms of G_1, G_2 and their corresponding open subgraphs G_1-e, G_2-e, is illustrated by Fig.5, where

$$R(G) = \frac{1}{p_2}\{(p_1p_2 + p_1p_4 + p_2p_4 - 2p_1p_2p_4)(p_2p_3 + p_2p_5 + p_3p_5 - 2p_2p_3p_5) - (1-p_2)(p_1p_4)(p_3p_5)\}.$$

G :

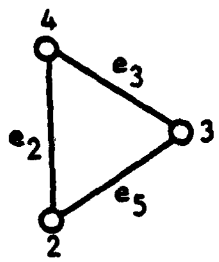


G_1 :



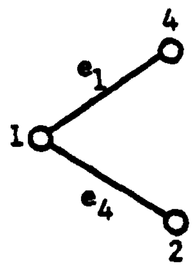
$R(G_1) = p_1p_2 + p_1p_4 + p_2p_4 - 2p_1p_2p_4$

G_2 :



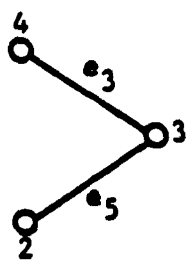
$R(G_2) = p_2p_3 + p_2p_5 + p_3p_5 - 2p_2p_3p_5$

$G_1 - e_2$:



$R(G_1 - e_2) = p_1p_4$

$G_2 - e_2$:



$R(G_2 - e_2) = p_3p_5$

FIGURE 5

Illustration for Theorem 3

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